2016 End of Year Lecture Exam
for 2016-2018 VCE study design

Engage Education Foundation

Units 3 and 4 Specialist Mathematics: Exam 2

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Any questions?

Check the Engage website for updated solutions, then email practiceexams@ee.org.au.

Section A – Multiple-choice questions

Question 1

The correct answer is A.

Find the intersections of the two functions using calculator; $1.305, 0, -1.305$. Both y_1 and y_2 are odd functions, so the area is given by $2\int_0^{1.305} y_1(x) - y_2(x)\, dx$

Question 2

The correct answer is D.

Store the function and its derivatives on your calculator. Evaluate each option.

Question 3

The correct answer is D.

$$\text{Time of flight} = \underbrace{2 \times \underbrace{\frac{3500 \sin 28^\circ}{9.8}}_{\substack{\text{time taken for projectile}\\ \text{to reach apex of arc}}}}_{\substack{\text{total time for projectile to traverse}\\ \text{parabolic arc, i.e. time until it is}\\ \text{20m above the ground again}}} + \underbrace{\underbrace{\frac{\sqrt{3500^2 + 4 \times 9.8 \times 20} - 3500}{2 \times 9.8}}_{\substack{\text{positive solution of } t \text{ for the equation } x=ut+at^2\\ \text{where } x=20\text{m},\ u=3500\sin 28^\circ \text{ms}^{-1} \text{ and } a=9.8 \text{ ms}^{-2}}}}_{\text{i.e.time taken for projectile to fall remaining 20m}}$$

Multiply time of flight by the horizontal component of velocity, $3500 \cos 28^\circ \text{ ms}^{-1}$.

Question 4

The correct answer is B.

$$\frac{1}{z} + \frac{1}{\bar{z}} = \frac{\bar{z} + z}{z\bar{z}} = \frac{2\,\mathrm{Re}(z)}{|z|^2} \in \mathbb{R}$$

Question 5

The correct answer is D.

Choose $u = \tan(x)$. Then $\frac{du}{dx} = \sec^2 x$. Evaluating terminals, $u\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $u\left(\frac{\pi}{6}\right) = 1/\sqrt{3}$, so

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \cot x = \int_{\frac{1}{\sqrt{3}}}^{\frac{\sqrt{3}}{2}} \sec^2 x \frac{1}{u} \frac{du}{\sec^2 x} = \int_{\frac{1}{\sqrt{3}}}^{\frac{\sqrt{3}}{2}} \frac{1}{u}\, du$$

Question 6

The correct answer is A.

$\boldsymbol{a} \times \Delta t = 3\boldsymbol{a} = \boldsymbol{v} - \boldsymbol{u} = 2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$. $|\boldsymbol{a}| = \sqrt{4 + 4 + 1} = 3$, so $\hat{\boldsymbol{a}} = \frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$

Question 7
The correct answer is B.

Forces acting are $\mathbf{1000N}$ and $-\boldsymbol{kv}$, net force is $\mathbf{1000} - \boldsymbol{kv}$. $\boldsymbol{a} = \frac{\boldsymbol{F}}{\boldsymbol{m}} = \frac{\mathbf{1000}-\boldsymbol{kv}}{\mathbf{800}} = \frac{\mathbf{5}}{\mathbf{4}} - \frac{\boldsymbol{kv}}{\mathbf{800}}$. Note that the resistance force is in Newtons.

Question 8
The correct answer is C.

$a = |\boldsymbol{u} \cdot \hat{\boldsymbol{v}}|$ and b $= |\boldsymbol{u} \cdot \hat{\boldsymbol{v}}|\hat{\boldsymbol{v}}$. So $\frac{\boldsymbol{b}}{a} = \hat{\boldsymbol{v}}$

Question 9
The correct answer is D.

Question 10
The correct answer is E.

Type II error indicates a false acceptance of the null hypothesis, i.e. a 'false negative'.

Question 11
The correct answer is A.

Either graph the function parametrically using your calculator, or make the observation that $(1 + 2\cos t)$ is a factor of both components and is zero at $\cos(t) = -\frac{1}{2}$, which is true when $t = \frac{2\pi}{3}, \frac{4\pi}{3}$

Question 12
The correct answer is D.

Look for key features of the slope field, in particular that the gradient is zero along both axes.

Question 13
The correct answer is D.

Question 14
The correct answer is A.

$\frac{d}{dx}\sec x = \frac{\sin(x)}{\cos^2(x)} = \sec(x)\tan(x)$. So $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec(x)\tan(x)e^{\sec(x)}dx = \int_{\sec(0)}^{\sec\left(\frac{\pi}{4}\right)} e^u\,du$, where $u = \sec x$.

Question 15
The correct answer is C.

$\overrightarrow{AC} \cdot \overrightarrow{CB} = 0$, as the angle subtended at the circumference by a diameter is $90°$
$\overrightarrow{AC} \cdot \left(-\overrightarrow{AC} + \overrightarrow{AB}\right) = 0$
$-\overrightarrow{AC} \cdot \overrightarrow{AC} + \overrightarrow{AC} \cdot \overrightarrow{AB} = 0$ using linearity of dot products
$\Rightarrow \left|\overrightarrow{AC}\right|^2 = \overrightarrow{AC} \cdot \overrightarrow{AB}$

Question 16
The correct answer is E.

Speed is the magnitude of the velocity vector.

$$\boldsymbol{r}(t) = \left(2\sqrt{2(t+1)} + 2\right)\boldsymbol{i} + \frac{\sqrt{2}}{2}\log_e(t+1)\,(\boldsymbol{j} - \boldsymbol{k})$$
$$\boldsymbol{r}'(t) = \frac{\sqrt{2}}{\sqrt{t+1}}\boldsymbol{i} + \frac{\sqrt{2}}{2(t+1)}(\boldsymbol{j} - \boldsymbol{k})$$
$$|\boldsymbol{r}'| = \sqrt{\frac{2}{t+1} + \frac{2}{4(t+1)^2} + \frac{2}{4(t+1)^2}}$$
$$|\boldsymbol{r}'| = \sqrt{\frac{2}{t+1} + \frac{1}{(t+1)^2}}$$
$$|\boldsymbol{r}'| = \frac{\sqrt{2t+3}}{t+1}$$

Question 17
The correct answer is C.

$V = \frac{4}{3}\pi r^3$, so $r = \sqrt[3]{\frac{3V}{4\pi}}$; hence $\frac{dr}{dV} = \left(\frac{1}{4\pi}\right)\left(\frac{3V}{4\pi}\right)^{-\frac{2}{3}}$

$0.2\pi \text{ Ls}^{-1} = 200\pi\ cm^3$, and 12 L is 12000 cm^3.

By the chain rule, $\frac{dr}{dt} = \frac{dr}{dV}\frac{dV}{dt} = \left(\frac{1}{4\pi}\right)\left(\frac{3V}{4\pi}\right)^{-\frac{2}{3}} 200\pi = 50\left(\frac{3V}{4\pi}\right)^{-\frac{2}{3}} = 0.25 \text{ cm/s}$

Question 18
The correct answer is C.

$\frac{dx}{dy} = \frac{2}{\sqrt{1-4y^2}} = \frac{1}{\sqrt{\frac{1}{4}-y^2}}$
$x = \sin^{-1}(2y) + c$
$\frac{1}{2} = \sin^{-1} 1 + c$ evaluating at the given point
$\Rightarrow c = \frac{1}{2} - \frac{\pi}{2}$
$\Rightarrow y = \frac{1}{2}\sin\left(x - \frac{1}{2} + \frac{\pi}{2}\right)$
$y = \frac{1}{2}\cos\left(x - \frac{1}{2}\right)$ as $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)$

Question 19
The correct answer is C.

Question 20
The correct answer is E.

$(x^3 - 1) = (x-1)(x^2 + x + 1)$, i.e. one linear factor and one irreducible quadratic factor

Question 21
The correct answer is C.

$a = \frac{F}{m} = \sin\left(\frac{3}{2}t\right)$
$v = \int a\,dt = -\frac{2}{3}\cos\left(\frac{3}{2}t\right) + c$
$1.5 + \frac{2}{3} = \frac{13}{6} = c$ finding a value for c using the initial condition
$x = \int v\,dt = -\frac{4}{9}\sin\left(\frac{3}{2}t\right) + \frac{13}{6}t$

Question 22

The correct answer is A.

To find Pr(3X > 2Y) is equivalent to finding Pr(3X – 2Y > 0).

In order to find Pr(3X – 2Y > 0), we have to consider the distribution of 3X – 2Y, which is normally distributed (due to the linear combination of independent normal distributions).

E(3X-2Y) = 3E(X) – 2E(Y) = -12.

$\sigma(3X - 2Y) = \sqrt{3^2 \times 3^2 + 2^2 \times 4^2} = \sqrt{145}$.

Hence, Pr(3X – 2Y > 0) can be found by using the CAS as follows

```
normCDf(0,1000000,145^(1/2),-12)
                    0.1594925229
```

Section B – Analysis

Marks allocated are indicated by a number in square brackets, for example, [1] indicates that the line is worth one mark.

Question 1a i

Using addition formulae,

$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)$

$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{1}{4}\left(\sqrt{6} - \sqrt{2}\right)$ [1]

$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

$\cos\left(\frac{\pi}{12}\right) = \frac{1}{2}\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} = \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$ [1]

as required.

Question 1a ii

Using a double angle formula:

$\cos\left(\frac{\pi}{12}\right) = 2\cos^2\left(\frac{\pi}{24}\right) - 1$

Rearranging,

$\cos\left(\frac{\pi}{24}\right) = \pm\sqrt{\frac{\cos\left(\frac{\pi}{12}\right)+1}{2}}$ we can discard the negative value as $0 \le \cos\left(\frac{\pi}{24}\right) \le 1$ [1]

$\cos\left(\frac{\pi}{24}\right) = \frac{\sqrt{\sqrt{6}+\sqrt{2}+4}}{2\sqrt{2}} = \frac{1}{2}\sqrt{\frac{1}{2}\left(\sqrt{6} + \sqrt{2} + 4\right)}$ [1]

Question 1b i

$z = \left(\frac{1}{2}\left(\sqrt{2} - \sqrt{6}\right), \frac{1}{2}\left(\sqrt{6} + \sqrt{2}\right)\right) = \left(-2\sin\left(\frac{\pi}{12}\right), 2\cos\left(\frac{\pi}{12}\right)\right)$ using part a.i. [1]

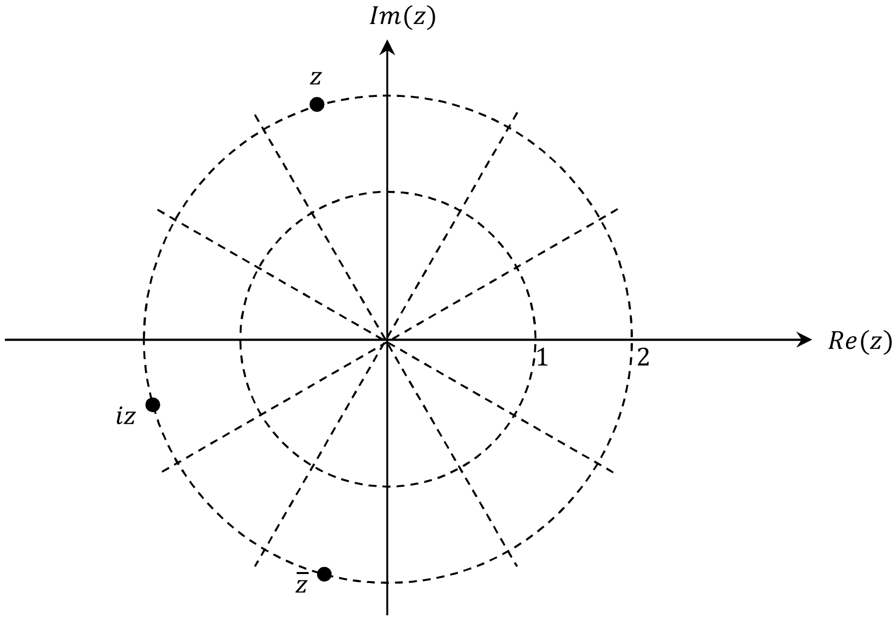
As $\cos\left(\frac{\pi}{2} + \frac{\pi}{12}\right) = -\sin\left(\frac{\pi}{12}\right)$ and $\sin\left(\frac{\pi}{2} + \frac{\pi}{12}\right) = \cos\left(\frac{\pi}{12}\right)$, $z = 2\,\text{cis}\left(\frac{7\pi}{12}\right)$ [1]

Therefore, using de Moivre's theorem, $z^5 = 32\,\text{cis}\left(\frac{35\pi}{12}\right) = 32\,\text{cis}\left(\frac{11\pi}{12}\right)$ [1]

Question 1b ii

For z^n to be purely imaginary, the real component must be zero. Therefore, the argument of z^n must be a multiple of π [1]. So we need all $n \in \mathbb{Z}$ such that $\frac{7\pi n}{12} = k\pi, k \in \mathbb{Z}$. Therefore n must be a multiple of 12 (as 7 and 12 are coprime) [1]. i.e. $n = 12m, m \in \mathbb{Z}$

Question 1b iii



$\bar{z}$ is given by a reflection of z in the x-axis. iz is given by a rotation of z by $\frac{\pi}{2}$

Question 2a i

$\Delta V = (\text{area of outer circle} - \text{area of inner circle}) \times \Delta y$
$= \pi \Delta y (r_2^2 - r_1^2)$

Question 2a ii

$y^2 + (x-2)^2 = 1$

Question 2a iii

Rearranging the equation for the circle in terms of x, we find that $x = 2 \pm \sqrt{1-y^2}$ [1]. For a given y, this equation gives the two corresponding x-values on the circle, which are r_1 and r_2. As $r_1 < r_2$, $r_1 = 2 - \sqrt{1-y^2}$ and $r_2 = 2 + \sqrt{1-y^2}$ [1]. Then

$\Delta V = \pi \Delta y \left(\left(2+\sqrt{1-y^2}\right)^2 - \left(2-\sqrt{1-y^2}\right)^2 \right)$
$\Delta V = \pi \Delta y \left(\left(4 + 4\sqrt{1-y^2} + (1-y^2)\right) - \left(4 - 4\sqrt{1-y^2} + (1-y^2)\right) \right)$
$\Delta V = 8\pi \Delta y \sqrt{1-y^2}$ [1]

Question 2b

By symmetry, we only need to calculate the volume of the top 3 washers, and then double that result to find the approximation. As the thickness of each washer is Δy, $3\Delta y = 1$, so $\Delta y = \frac{1}{3}$ [1]. As the circles pass halfway through the edges of the washers, the washers, from top to bottom, have y-values $\frac{5}{6}$, $\frac{1}{2}$ and $\frac{1}{6}$ [1]. Therefore,

$\frac{1}{2} V_{approx} = \frac{8\pi}{3}\sqrt{1-\left(\frac{5}{6}\right)^2} + \frac{8\pi}{3}\sqrt{1-\left(\frac{1}{2}\right)^2} + \frac{8\pi}{3}\sqrt{1-\left(\frac{1}{6}\right)^2}$
$V_{approx} = \frac{16\pi}{3}\left(\frac{\sqrt{11}}{6} + \frac{\sqrt{3}}{2} + \frac{\sqrt{35}}{6}\right)$
$V_{approx} = 40.29 \text{ cubic units}$ [1]

Question 2c i

From part a.iii., $\Delta V = 8\pi\Delta y\sqrt{1-y^2}$. Therefore,

$\frac{\Delta V}{\Delta y} = 8\pi\sqrt{1-y^2}$
$\Rightarrow \frac{dV}{dy} = 8\pi\sqrt{1-y^2}$

Integrating with respect to y between $y = -1$ and $y = 1$,

$V = 8\pi \int_{-1}^{1} \sqrt{1-y^2}\, dy$

Question 2c ii.

$\frac{d}{dy}\left(y\sqrt{1-y^2} + \sin^{-1}(y)\right) = \sqrt{1-y^2} - \frac{y^2}{\sqrt{1-y^2}} + \frac{1}{\sqrt{1-y^2}} = 2\left(\frac{1-y^2}{\sqrt{1-y^2}}\right) = 2\sqrt{1-y^2}$ [2]

Question 2c iii

$V = 8\pi \int_{-1}^{1} \sqrt{1-y^2}\, dy$
$V = \frac{8\pi}{2}\left[y\sqrt{1-y^2} + \sin^{-1}(y)\right]_{-1}^{1}$ [1]
$V = 4\pi\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$
$V = 4\pi^2$ [1]

as required

Question 3a

$a = v\frac{dv}{dx} = -\frac{1}{24}(2v^2+5)$
$\Rightarrow \frac{dx}{dv} = -\frac{24v}{2v^2+5}$
$x = \int -\frac{24v}{2v^2+5}\, dv$

Let $u = 2v^2 + 5$. Then $\frac{du}{dv} = 4v$. So,
$x = \int -\frac{24v}{u}\frac{1}{4v}\, du$
$x = -6\log_e(2v^2+5) + c$ [1]

We know that at the moment the rocket is launched, $x = 0$ and $v = 50$. Substituting, gives $c = 6\log_e 5005$. Therefore,

$x = 6\log_e \frac{5005}{2v^2+5}$ [1]

When the rocket has reached the apex of it's flight, $v = 0$. Evaluating,

$x_{max} = 6\log_e 1001 \approx 41.45 \text{ m}$ [1]

Question 3b i

$a = \frac{dv}{dt} = -\frac{1}{24}(2v^2 + 5)$

$\Rightarrow \frac{dt}{dv} = -\frac{24}{2v^2+5}$

$t = -\frac{12}{\sqrt{\frac{5}{2}}} \int \frac{\sqrt{\frac{5}{2}}}{v^2+\frac{5}{2}} dv$

$t = -\frac{12\sqrt{10}}{5} \tan^{-1}\left(\frac{v\sqrt{10}}{5}\right) + c$ [1]

As $v = 50$ when $t = 0$, $c = \frac{12\sqrt{10}}{5} \tan^{-1}\left(10\sqrt{10}\right)$ [1].

Substituting:

$t = \frac{12\sqrt{10}}{5}\left(\tan^{-1}\left(10^{\frac{3}{2}}\right) - \tan^{-1}\left(\frac{v\sqrt{10}}{5}\right)\right)$

Question 3b ii

$v = 0$ at the apex, so,

$t = \frac{12\sqrt{10}}{5} \tan^{-1}\left(10^{\frac{3}{2}}\right)$ [1]

Question 3c i

$$\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)} dx$$

Let $u = \cos(x)$. Then $\frac{du}{dx} = -\sin(x)$. So [1],

$\int \tan(x) = \int \frac{\sin(x)}{u} \frac{1}{-\sin(x)} du$

$\int \tan(x) = -\log_e|u| + c = \log_e|\cos(x)| + c$ [1] as required.

Question 3c ii

We first need to rearrange the equation from b.i. to yield an equation for v in terms of t

$$t = \frac{12\sqrt{10}}{5}\left(\tan^{-1}\left(10^{\frac{3}{2}}\right) - \tan^{-1}\left(\frac{v\sqrt{10}}{5}\right)\right)$$

Rearranging,

$$\tan^{-1}\left(10^{\frac{3}{2}}\right) - \frac{5t}{12\sqrt{10}} = \tan^{-1}\left(\frac{v\sqrt{10}}{5}\right)$$
$$\frac{v\sqrt{10}}{5} = \tan\left(\tan^{-1}\left(10^{\frac{3}{2}}\right) - \frac{\sqrt{10}}{24}t\right)$$
$$v = \frac{\sqrt{10}}{2}\tan\left(\tan^{-1}\left(10^{\frac{3}{2}}\right) - \frac{\sqrt{10}}{24}t\right) \text{ [1]}$$

It is tempting to use an addition formula, however doing so gives $v = \frac{\sqrt{10}}{2}\left(\frac{10^{\frac{3}{2}} - \tan\left(\frac{\sqrt{10}}{24}t\right)}{1 + 10^{\frac{3}{2}}\tan\left(\frac{\sqrt{10}}{24}t\right)}\right)$ which is much harder to integrate.

$$x = \int v\,dt = \frac{\sqrt{10}}{2}\int \tan\left(\tan^{-1}\left(10^{\frac{3}{2}}\right) - \frac{\sqrt{10}}{24}t\right)\,dt$$

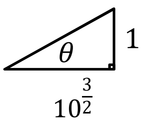
Let $u = \tan^{-1}\left(10^{\frac{3}{2}}\right) - \frac{\sqrt{10}}{24}t$. Then $\frac{du}{dt} = -\frac{\sqrt{10}}{24}$. So,

$$x = \frac{\sqrt{10}}{2}\int \tan(u)\frac{-24}{\sqrt{10}}\,dt \text{ [1]}$$

Using part c.i.

$$x = -12\left(-\log_e\left|\cos\left(\tan^{-1}\left(10^{\frac{3}{2}}\right) - \frac{\sqrt{10}}{24}t\right)\right| + c\right)$$
$$x = 12\log_e\left|\cos\left(\tan^{-1}\left(10^{\frac{3}{2}}\right) - \frac{\sqrt{10}}{24}t\right)\right| - 12c \quad \text{[1]}$$

As $x = 0$ when $t = 0$, $c = \log_e\left|\cos\left(\tan^{-1}\left(10^{\frac{3}{2}}\right)\right)\right|$



Consider the triangle shown above, which is constructed so that $\theta = \tan^{-1}\left(10^{\frac{3}{2}}\right)$.

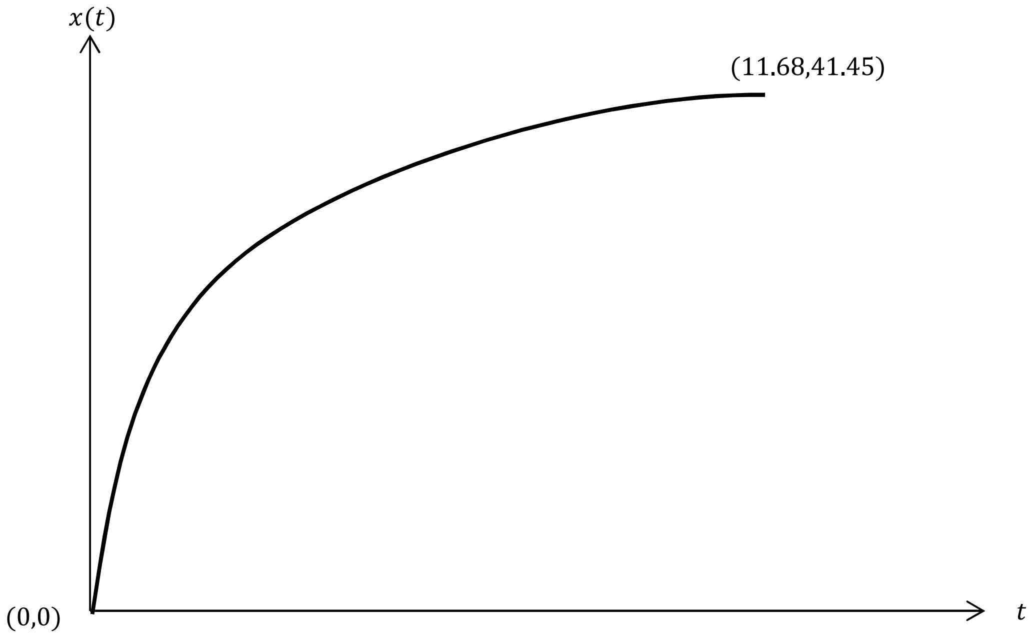
The hypotenuse is $\sqrt{1001}$. So $\cos\theta = \frac{1}{\sqrt{1001}}$. Therefore $c = -\log_e\sqrt{1001}$.

Finally,

$$x = 12\log_e\left|\cos\left(\tan^{-1}\left(10^{\frac{3}{2}}\right) - \frac{\sqrt{10}}{24}t\right)\right| + 12\log_e\sqrt{1001}$$
$$x = 12\log_e\left(\sqrt{1001}\left|\cos\left(\tan^{-1}\left(10^{\frac{3}{2}}\right) - \frac{\sqrt{10}}{24}t\right)\right|\right), \ 0 \le t \le \frac{12\sqrt{10}}{5}\tan^{-1}\left(10^{\frac{3}{2}}\right) \text{ [1] as required.}$$

Question 3c iii

Correct shape [1]. Correct points [1]



Question 4a

$\frac{dZ}{dt} = \alpha Z$ and $\frac{dL}{dt} = -\delta L$ where α and δ are positive constants.

Question 4b i

$\frac{dZ}{dt} = \alpha Z - \beta LZ$ is equivalent to $\frac{1}{Z}\frac{dZ}{dt} = \alpha - \beta L$, which is equivalent to $\frac{1}{\alpha-\beta L}\frac{1}{Z}\frac{dZ}{dt} = 1$ [1/2]

$\frac{dL}{dt} = -\gamma L + \delta LZ$ is equivalent to $\frac{1}{L}\frac{dL}{dt} = -\gamma + \delta Z$, which is equivalent to $\frac{1}{-\gamma+\delta Z}\frac{1}{L}\frac{dL}{dt} = 1$ [1/2]

Therefore,

$\frac{1}{-\gamma+\delta Z}\frac{1}{L}\frac{dL}{dt} - \frac{1}{\alpha-\beta L}\frac{1}{Z}\frac{dZ}{dt} = 0$

$\left(\frac{1}{-\gamma+\delta Z}\frac{1}{L}\frac{dL}{dt} - \frac{1}{\alpha-\beta L}\frac{1}{Z}\frac{dZ}{dt}\right)(\alpha - \beta L)(-\gamma + \delta Z) = 0$

$\frac{\alpha-\beta L}{L}\frac{dL}{dt} + \frac{\gamma-\delta Z}{Z}\frac{dZ}{dt} = 0$ [1] as required.

Question 4b ii

$\frac{\gamma-\delta Z}{Z}\frac{dZ}{dt} = \frac{d}{dt}\left(\int \frac{\gamma-\delta Z}{Z} dZ\right) = \frac{d}{dt}\left(\gamma \int \frac{1}{Z} dZ - \delta \int 1 dZ\right) = \frac{d}{dt}(\gamma \log_e Z - \delta Z)$ [1]

$\frac{\alpha-\beta L}{L}\frac{dL}{dt} = \frac{d}{dt}\left(\int \frac{\alpha-\beta L}{L} dL\right) = \frac{d}{dt}\left(\alpha \int \frac{1}{L} dL - \beta \int 1 dL\right) = \frac{d}{dt}(\alpha \log_e L - \beta L)$ [1]

Therefore,

$\frac{d}{dt}(\gamma \log_e Z - \delta Z + \alpha \log_e L - \beta L) = 0$

So,

$\gamma \log_e Z - \delta Z + \alpha \log_e L - \beta L = c$ [1]

Question 4c

Evaluating $0.9\log_e Z - 0.005Z + 0.6\log_e L - 0.02L = c$ when $Z = 100$ and $L = 40$ gives $c = 5.058$

Question 5

a. Null hypothesis: the mean viral load of the treated population is the same as the mean viral load of the general population of patients.
Alternative hypothesis: the mean viral load of the treated population is different from the mean viral load of the general population of patients.
[1] for both correct statements. Students can also use symbols/notations but should define the pronumerals before use.

b. For the tested sample group:
$\bar{x} = 950000$ and $sd(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{100000}{\sqrt{20}}$ [1]
Under the null hypothesis: $\mu = 1000000$.
Hence, under the distribution where $E(\bar{X}) = \mu = 1000000$ and $sd(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{60000}{\sqrt{120}}$,
p-value $= 2\Pr(\bar{X} < 950000|\mu = 1000000) = 2\Pr\left(Z < \frac{950000-1000000}{\frac{100000}{\sqrt{20}}}\right)$
$= 2\Pr(Z < -2.23607)$ (found from inverse cdf using CAS)

```
normCDf(-100000000,-2.23607,1,0)
                         0.01267359311
2ans
                         0.02534718622
```

$= 2 \times 0.01267$
$= 0.025$ [1]

c. P-value < 0.05. Hence, the conclusion is that the null hypothesis that the mean viral load of the treated population is the same as the mean viral load of the general population of patients is rejected [1].

d. If the null hypothesis is not rejected, p-value must be equal to or higher than α.
This means p-value≥ 0.05
$\therefore 2\Pr(\bar{X} < 950000|\mu = 1000000) \geq 0.05$

$$\therefore \Pr\left(Z < \frac{950000 - 1000000}{\frac{100000}{\sqrt{n}}}\right) \geq 0.025$$

$$\therefore \frac{950000 - 1000000}{\frac{100000}{\sqrt{n}}} \geq -1.96$$

```
invNormCDf("L",0.025,1,0)
                         -1.959963985
```

For a maximum value of n

$$\therefore \frac{950000 - 1000000}{\frac{100000}{\sqrt{n}}} = -1.96$$

Solving for n, we get

$$\text{solve}\left(\frac{950000-1000000}{\frac{100000}{\sqrt{x}}}=-1.96, x\right)$$

$$\{x=15.3664\}$$

Hence, the sample size can be 15 at most [1].

Question 6a i

If the plane maintains a constant altitude, then $|r(t)|$ (i.e. its distance from the centre of the earth) should be constant [1]

$\frac{|r(t)|^2}{h^2} = \cos^2 t \cos^2 a + \sin^2 t \cos^2 a + \sin^2 a$

$\frac{|r(t)|^2}{h^2} = \cos^2 a\,(\cos^2 t + \sin^2 t) + \sin^2 a$

$\frac{|r(t)|^2}{h^2} = \cos^2 a + \sin^2 a$

$|r(t)|^2 = h^2$, so $|r(t)| = h$ [1]

Question 6a ii

$h = 6378100 + 13700 = 6391800$ [1]

Question 6b i

If $a = \tan^{-1}\left(\frac{t}{2}\right)$, then $\cos(a) = \frac{2}{\sqrt{t^2+4}}$ and $\sin(a) = \frac{t}{\sqrt{t^2+4}}$

So $\boldsymbol{r}(t) = \frac{2h\cos(t)}{\sqrt{t^2+4}}\boldsymbol{i} + \frac{ht\sin(t)}{\sqrt{t^2+4}}\boldsymbol{j} + \frac{ht}{\sqrt{t^2+4}}\boldsymbol{k}$. Therefore,

$$\dot{\boldsymbol{r}}(t) = \frac{-2h\left(\sqrt{t^2+4}\sin(t)+t\left(t^2+4\right)^{-\frac{1}{2}}\cos(t)\right)}{t^2+4}\boldsymbol{i} + \frac{h\left(\sqrt{t^2+4}(\sin(t)+t\cos(t))+t^2\left(t^2+4\right)^{-\frac{1}{2}}\sin(t)\right)}{t^2+4}\boldsymbol{j} + \frac{h\left(\sqrt{t^2+4}-t^2\left(t^2+4\right)^{-\frac{1}{2}}\right)}{t^2+4}\boldsymbol{k}$$

$$\dot{\boldsymbol{r}}(t) = \frac{-2h\left((t^2+4)\sin(t)+t\cos(t)\right)}{(t^2+4)^{\frac{3}{2}}}\boldsymbol{i} + \frac{h\left((t^2+4)(\sin(t)+t\cos(t))+t\sin(t)\right)}{(t^2+4)^{\frac{3}{2}}}\boldsymbol{j} + \frac{4h}{(t^2+4)^{\frac{3}{2}}}\boldsymbol{k}$$

$$\dot{\boldsymbol{r}}(t) = \frac{-2h\left((t^2+4)\sin(t)+t\cos(t)\right)}{(t^2+4)^{\frac{3}{2}}}\boldsymbol{i} + \frac{h\left(t(t^2+4)\cos(t)+4\sin(t)\right)}{(t^2+4)^{\frac{3}{2}}}\boldsymbol{j} + \frac{4h}{(t^2+4)^{\frac{3}{2}}}\boldsymbol{k} \quad [2]$$

where h is as defined above.

Question 6b ii

$\dot{\boldsymbol{r}}(2) = 3905.28\ \text{km/h}$