Engage Education Foundation

Units 3 and 4 Specialist Mathematics: Exam 1

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Any questions?

Check the Engage website for updated solutions, then email practiceexams@ee.org.au.

Marks allocated are indicated by a number in square brackets, for example, [1] indicates that the line is worth one mark.

Question 1a

$(iz)^6 = 1$

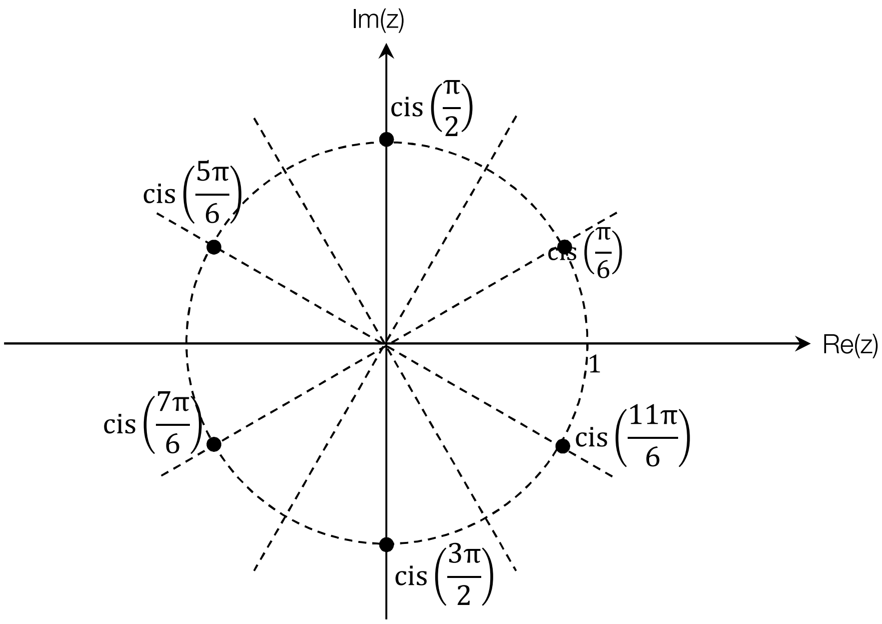
$\Rightarrow i^6 z^6 = 1$

$\Rightarrow z^6 = -1 = \text{cis}\,(\pi + 2\pi k), k \in \mathbb{Z}$

$\Rightarrow z = \text{cis}\left(\frac{\pi}{6} + \frac{\pi k}{3}\right)$ [1] by De Moivre's Theorem

$z = \text{cis}\left(\frac{\pi}{6}\right), \text{cis}\left(\frac{\pi}{2}\right), \text{cis}\left(\frac{5\pi}{6}\right), \text{cis}\left(\frac{7\pi}{6}\right), \text{cis}\left(\frac{3\pi}{2}\right), \text{cis}\left(\frac{11\pi}{6}\right)$ [2]

Question 1b



Question 2

$\left(\frac{z}{2}\right)^2 = \left(\frac{3}{\bar{z}}\right)^2$

$\Rightarrow (z\bar{z})^2 = 36$

$\Rightarrow |z| = 6$

i.e. a circle of radius 6

Question 3a

$y = \log_e(\tan(x) + \sec(x))$

$\frac{dy}{dx} = \frac{du}{dx}\frac{d}{du}\log_e u$, where $u = \tan(x) + \sec(x)$ [1]

$\frac{d}{du}\log_e u = \frac{1}{u}$

$\frac{du}{dx} = \frac{d}{dx}(\tan(x)) + \frac{d}{dx}\left(\frac{1}{\cos(x)}\right)$

$\frac{du}{dx} = \frac{\cos(x)\cos(x) - (-\sin(x))\sin(x)}{\cos(x)^2} + \left(-\frac{(-\sin(x))}{\cos^2(x)}\right)$ by product rule and chain rule, respectively [1]

$\frac{du}{dx} = \sec^2(x) + \sec(x)\tan(x)$ [1]

$\frac{dy}{dx} = \frac{\sec^2(x) + \sec(x)\tan(x)}{\tan(x) + \sec(x)} = \frac{\sec(x) + \tan(x)}{\sin(x) + 1} = \sec(x)\left(\frac{1 + \sin(x)}{\sin(x) + 1}\right) = \sec(x)$ [1]

Question 3b i

$1 = \sin^2(x) + \cos^2(x)$ [1]

$1 - \sin^2(x) = \cos^2(x)$

$(1 - \sin(x))(1 + \sin(x)) = \cos^2(x)$

$1 - \sin(x) = \frac{\cos^2(x)}{1+\sin(x)}$, as required [1]

Question 3b ii

$\sqrt{\frac{1-\sin(x)}{1+\sin(x)}} = \sqrt{\frac{\cos^2(x)}{(1+\sin(x))^2}} = \frac{\cos(x)}{1+\sin(x)}$, using (i) [1]. We do not need an absolute value as $\cos x$ and $\sin x$ are non-negative on the interval $[0, \frac{\pi}{2}]$.

Let $u = 1 + \sin(x)$. Then $\frac{du}{dx} = \cos(x)$, so

$\int \sqrt{\frac{1-\sin(x)}{1+\sin(x)}}\, dx = \int \frac{\cos(x)}{u} \frac{1}{\cos(x)}\, du = \log_e(\sin(x) + 1)$ [1]

Question 4

$let\ u = 2x + y, \frac{du}{dx} = 2 + \frac{dy}{dx}$

$e^y \frac{dy}{dx} = \cos(2x + y)\left(2 + \frac{dy}{dx}\right)$[1]

$e^y \frac{dy}{dx} - \frac{dy}{dx}\cos(2x + y) = 2\cos(2x + y)$

$\frac{dy}{dx} = \frac{2\cos(2x+y)}{e^y - \cos(2x+y)}$ [1]

Question 5

$\frac{y^2}{4} - \frac{(x+1)^2}{3} = 1$

$\frac{dy}{dx}\frac{y}{2} - \frac{2(x+1)}{3} = 0$

$\frac{dy}{dx} = \frac{4(x+1)}{3y}$ [1]

When $x = 2$, $\frac{y^2}{4} - 3 = 1$, so $y = \pm 4$. As we want the bottom half of the curve, $y = -4$ [1].

The gradient at $(2, -4)$ is then $\frac{dy}{dx} = \frac{4(3)}{-12} = -1$ [1].

Therefore, the gradient of the normal will be 1. Then evaluating $y_{norm} = x + c$ at $(2, -4)$ gives $c = -6$, so the normal is given by $y_{norm} = x - 6$ [1]

Question 6a

$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$

$\sin\pi = 5\sin\frac{\pi}{5} - 20\sin^3\frac{\pi}{5} + 16\sin^5\frac{\pi}{5}$ [1]

$0 = \sin\frac{\pi}{5}\left(5 - 20\sin^2\frac{\pi}{5} + 16\sin^4\frac{\pi}{5}\right)$

Now $\sin\frac{\pi}{5} \neq 0$, so $\left(5 - 20\sin^2\frac{\pi}{5} + 16\sin^4\frac{\pi}{5}\right)$, which is a quadratic in $\sin^2\frac{\pi}{5}$, must be zero. Solving via the quadratic formula, [1]

$\sin^2\frac{\pi}{5} = \frac{20\pm\sqrt{400-320}}{32} = \frac{5\pm\sqrt{5}}{8}$

$\Rightarrow \sin\left(\frac{\pi}{5}\right) = \sqrt{\frac{5\pm\sqrt{5}}{8}}$ [1]

As $\frac{5}{8} > \frac{1}{2}$, $\sqrt{\frac{5}{8}} > \sqrt{\frac{1}{2}} = \sin\frac{\pi}{4}$. But $\sin\frac{\pi}{4} > \sin\frac{\pi}{5}$, so $\sin\frac{\pi}{5} < \sqrt{\frac{5}{8}}$. Therefore, $\sin\left(\frac{\pi}{5}\right) = \sqrt{\frac{5-\sqrt{5}}{8}}$ [1]

Question 6b

$\cos\frac{\pi}{5} = \sqrt{1 - \sin^2\frac{\pi}{5}} = \sqrt{\frac{3+\sqrt{5}}{8}}$ [1]

$\cos\frac{4\pi}{5} = \cos\left(\pi - \frac{\pi}{5}\right) = -\cos\frac{\pi}{5} = -\sqrt{\frac{3+\sqrt{5}}{8}}$ [1]

Question 7

a. $E(2X - Y + 3Z + 1) = 18$

$\therefore 2E(X) - E(Y) + 3E(Z) + 1 = 18$ [1]

$\therefore E(Z) = \frac{18-2\times9+7-1}{3} = 2$ [1].

b. $Var(2X - 3Y - 2Z + 6) = 293$

$\therefore 4Var(X) + 9Var(Y) + 4Var(Z) = 293$

$\therefore Var(Y) = \frac{293-4\times8-4\times9}{9} = 25$ [1]

$\therefore sd(Z) = \sqrt{Var(Y)} = 5$ [1]

Question 8a

$V = \int_0^k \pi x^2 dy$ [1]

$= \pi\int_0^k ye^{-2\pi y^2} dy$

Let $u = -2\pi y^2$, then $\frac{du}{dy} = -4\pi y$. So,

$V = \pi\int_0^{-2\pi k^2} -\frac{1}{4\pi y} ye^u du$

$= \frac{1}{4}\int_{-2\pi k^2}^{0} e^u du$

$= \frac{1}{4}\left(1 - e^{-2\pi k^2}\right)$ [1]

Question 8b

We need to find $\frac{dk}{dt} = \frac{dk}{dV}\frac{dV}{dt}$ [1]

$V = \frac{1}{4}\left(1 - e^{-2\pi k^2}\right)$

$\frac{dV}{dk} = -\frac{\pi}{2}e^{-2\pi k^2}$[1]

$\frac{dk}{dt} = -\frac{\pi}{2}e^{-2\pi k^2} \times 0.1$ [1]

Question 9a

$v = \frac{2}{x\sqrt{3+x^2}}$

$\frac{1}{2}v^2 = \frac{2}{3x^2+x^4}$

$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{6x+4x^3}{(3x^2+x^4)^2}$

Question 9b

$\frac{dx}{dt} = \frac{2}{x^2\sqrt{3+x^3}}$

$\frac{dt}{dx} = \frac{1}{2}x^2\sqrt{3+x^3}$

$t = \frac{1}{2}\int x^2\sqrt{3+x^3}\,dx$ [1]

$t = \frac{1}{2}\left(x^2 \times \frac{1}{3x^2} \times \frac{2}{3}(3+x^3)^{\frac{3}{2}} + c\right)$

$t = \frac{1}{9}((3+x^3)^{\frac{3}{2}} + c)$

As $t = 0$ when $x = 0$, $c = -3^{\frac{3}{2}}$ [1].

Therefore, $(3+x)^{\frac{3}{2}} = 9t + 3^{\frac{3}{2}}$

$x = \left(\left(9t + 3^{\frac{3}{2}}\right)^{\frac{2}{3}} - 3\right)^{\frac{1}{3}}$ [1]