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Free Exam
for 2006-15 VCE study design

Engage
Education
Foundation

Units 3 and 4 Specialist Maths: Exam 1

Practice Exam Solutions

Stop!

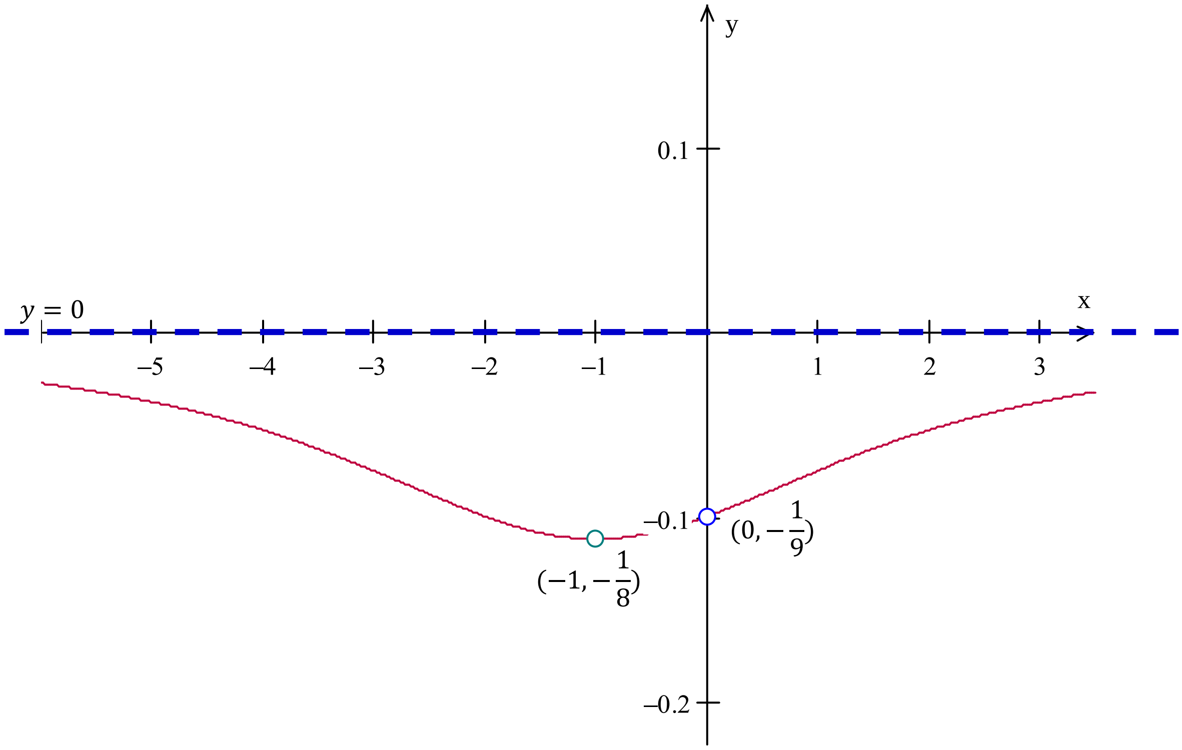
Don't look at these solutions until you have attempted the exam.

Any questions?

Check the Engage website for updated solutions, then email practiceexams@ee.org.au.

Section A – Multiple-choice questions

Question 1



shape of graph [1]

$f(0) = \frac{-1}{0^2+2\times 0+9} = \frac{-1}{9}$ correctly labelled intercept [1]

$f'(x) = \frac{d}{dx}(-(x^2+2x+9)^{-1})$
$= -1 \times -1 \times (x^2+2x+9)^{-2} \times (2x+2)$
$= \frac{2x+2}{(x^2+2x+9)^2}$
$Let\ f'(x) = 0$
$(x^2+2x+9)^{-2} \neq 0, \therefore 2x+2 = 0$
Hence $x = -1$ is a stationary point. correctly labelled stationary point [1]

$f(-1) = \frac{-1}{(-1)^2+2\times -1+9} = \frac{-1}{8}$
As $x \to \infty, f(x) \to 0$
Hence $y = 0$ is an asymptote of $f(x)$. correctly labelled asymptotes [1]

$x^2+2x+9 \neq 0$ so there are no vertical asymptotes.

Question 2

Show $\overrightarrow{OA} = \overrightarrow{CB}$ and $\overrightarrow{AB} = \overrightarrow{OC}$. Opposite sides are the same vector, so the same length and parallel)

stating requirements of a parallelogram [1]

$\overrightarrow{OA} = 2\boldsymbol{i} + 5\boldsymbol{j}$
$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$
$= 7\boldsymbol{i} + 6\boldsymbol{j} - (5\boldsymbol{i} + \boldsymbol{j})$
$= 2\boldsymbol{i} + 5\boldsymbol{j}$
$= \overrightarrow{OA}$ — showing $\overrightarrow{OA} = \overrightarrow{CB}$ [1]

$\overrightarrow{OC} = 5\boldsymbol{i} + \boldsymbol{j}$
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= (7\boldsymbol{i} + 6\boldsymbol{j}) - (2\boldsymbol{i} + 5\boldsymbol{j})$
$= 5\boldsymbol{i} + \boldsymbol{j}$
$= \overrightarrow{OC}$ — showing $\overrightarrow{AB} = \overrightarrow{OC}$ [1]

Therefore OABC is a parallelogram.

Question 3

$\frac{d}{dx}(x^2y + \log_e y + x) = \frac{d}{dx}(2)$

$2xy + x^2\frac{dy}{dx} + \frac{1}{y} \times \frac{dy}{dx} + 1 = 0$ — good attempt at implicit differentiation [1]

$\frac{dy}{dx}\left(x^2 + \frac{1}{y}\right) = -1 - 2xy$

$\therefore \frac{dy}{dx} = \frac{-1-2xy}{x^2+\frac{1}{y}}$ — correct answer [1]

Question 4

$P(z) = z^2 + 5z + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 8$

$= \left(z + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{8 * 4}{4}$

$= \left(z + \frac{5}{2}\right)^2 + \frac{7}{4}$ — successful completion of the square [1]

$= \left(z + \frac{5}{2}\right)^2 - \left(\frac{\sqrt{7}}{2}i\right)^2$ — (other methods also ok, eg. quadratic formula)

$= \left(z + \frac{5}{2} + \frac{\sqrt{7}}{2}i\right)\left(z + \frac{5}{2} - \frac{\sqrt{7}}{2}i\right)$

$= 0$

So, by the null factor theorem,

$z + \frac{5}{2} + \frac{\sqrt{7}}{2}i = 0 \; or \; z + \frac{5}{2} - \frac{\sqrt{7}}{2}i = 0$

$\therefore z = -\frac{5}{2} \pm \frac{\sqrt{7}}{2}i$ — both answers correct [2]

Question 5

$area = \left|\int_{-2}^{-1} \frac{7x+1}{x^2+2x-8} dx\right|$ correct integral for area [1]

$$\frac{7x+1}{x^2+2x-8} = \frac{7x+1}{(x+4)(x-2)} \equiv \frac{A}{x+4} + \frac{B}{x-2}$$

$7x + 1 \equiv A(x-2) + B(x+4)$

Let $x = -4$

$7 \times -4 + 1 = A(-4-2) + B(-4+4)$

$\therefore A = \frac{9}{2}$

Let $x = 2$

$7 \times 2 + 1 = A(2-2) + B(2+4)$

$\therefore B = \frac{5}{2}$

$\Rightarrow \int_{-2}^{-1} \frac{7x+1}{x^2+2x-8} dx = \int_{-2}^{-1} \frac{99}{2(x+4)} + \frac{5}{2(x-2)} dx$ correct splitting of fraction [1]

$= \left[\frac{9}{2}\log_e|x+4| + \frac{5}{2}\log_e|x-2|\right]_{-2}^{-1}$ correct integration [1]

$= \left[\frac{9}{2}\log_e|-1+4| + \frac{5}{2}\log_e|-1-2|\right] - \left[\frac{9}{2}\log_e|-2+4| + \frac{5}{2}\log_e|-2-2|\right]$

$= \frac{9}{2}\log_e(3) + \frac{5}{2}\log_e(3) - \frac{9}{2}\log_e(2) - \frac{5}{2}\log_e(4)$

$= 7\log_e(3) - \frac{9}{2}\log_e(2) - \frac{5}{2}\log_e(4)$ answer [1]

Question 6a

$x = \sec^2(t) \ and \ y^2 = \tan^2(t)$

Using the trigonometric identity $1 + \tan^2(x) = \sec^2(x)$: [1]

$1 + y^2 = x$

$y^2 = x - 1$

$\therefore y = \pm\sqrt{x-1}$ but $y > 0$ in the provided domain,

so y = $\sqrt{x-1}$ answer [1]

Question 6b

$x = sec^2(t)$ for $t \in \left\{t: 0 \le t \le \frac{\pi}{4}\right\}$

$\sec^2(0) = \left(\frac{1}{\cos(0)}\right)^2 = 1$

$\sec^2\left(\frac{\pi}{4}\right) = \left(\frac{1}{\cos\left(\frac{\pi}{4}\right)}\right)^2 = 2$

Hence $1 \le x \le 2$.

$y = \tan(t)$ for $t \in \left\{t: 0 \le t \le \frac{\pi}{4}\right\}$

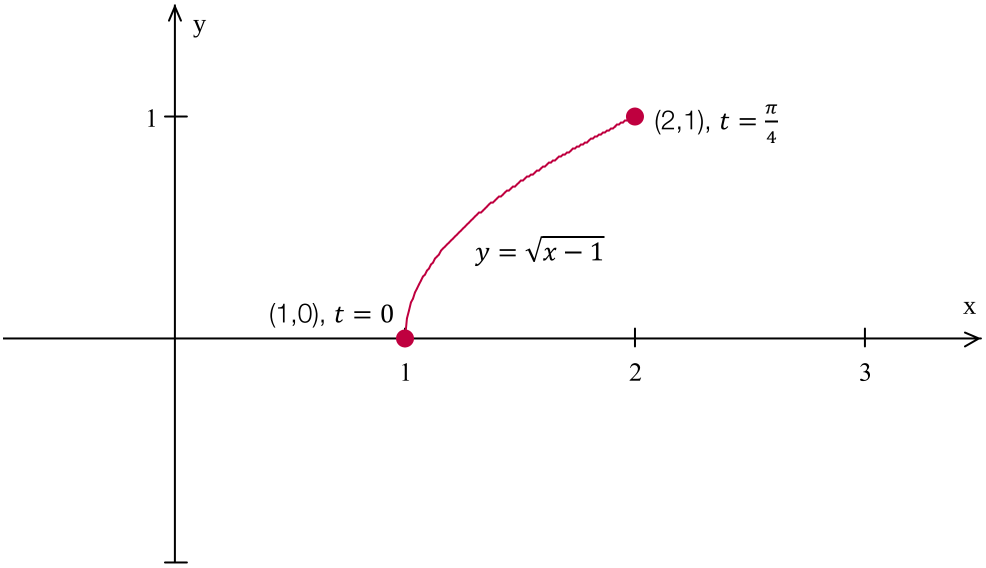
$\tan(0) = 0$

$\tan\left(\frac{\pi}{4}\right) = 1$

Hence $0 \le y \le 1$.

$dom = [1,2]$ and $ran = [0,1]$ answers [2]

Question 6c



graph shape [1], intercept with coordinate and t value [1], closed endpoints with coordinates and t values [1]

Question 7

$cosec\left(\frac{5\pi}{12}\right) = \frac{1}{\sin\left(\frac{5\pi}{12}\right)}$ use of reciprocal circular identities [1]

$\frac{5\pi}{12} = \frac{4\pi}{12} + \frac{\pi}{12} = \frac{4\pi}{12} + \left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{3} + \frac{\pi}{3} - \frac{\pi}{4} = \frac{2\pi}{3} - \frac{\pi}{4}$

$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\therefore \sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$ correct selection of double angle formula [1]

$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

$\therefore cosec\left(\frac{5\pi}{12}\right) = \frac{4}{\sqrt{6}+\sqrt{2}}$

$= \frac{4(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}$

$= \frac{4(\sqrt{6}-\sqrt{2})}{6-2}$ working leading to correct answer [1]

$= \frac{4(\sqrt{6}-\sqrt{2})}{4} = \sqrt{6}-\sqrt{2}$, as required.

Question 8

$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{5}\right)\right) = \frac{5}{25+x^2}$

Let $u = \tan^{-1}\left(\frac{x}{5}\right), \frac{du}{dx} = \frac{5}{25+x^2}$ correct substitution [1]

$\int \frac{\tan^{-1}\frac{x}{5}}{25+x^2}dx = \frac{1}{5}\int \tan^{-1}\frac{x}{5} * \frac{5}{25+x^2}dx$

$= \frac{1}{5}\int u\frac{du}{dx}dx$

$= \frac{1}{5}\int udu$ successful working [1]

$\frac{1}{5} \times \frac{1}{2}u^2 + c$ where $c \in \mathbb{R}$

$\therefore \int \frac{\tan^{-1}\frac{x}{5}}{25+x^2}dx = \frac{1}{10}\left(\tan^{-1}\frac{x}{5}\right)^2 + c$ answer [1]

Question 9

From formula sheet:

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\Rightarrow \cos^2(x) = \frac{\cos(2x)+1}{2}$ and $\sin^2(x) = \frac{1-\cos(2x)}{2}$

$\therefore \int_0^\pi cos^2(x)sin^2(x)dx = \int_0^\pi \left(\frac{\cos(2x)+1}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)dx$

$= \frac{1}{4}\int_0^\pi 1 - \cos^2(2x)\,dx$ correct use of double-angle formula [1]

$\cos(4x) = 2\cos^2(2x) - 1 \Rightarrow \cos^2(2x) = \frac{1+\cos(4x)}{2}$

$\therefore \frac{1}{4}\int_0^\pi 1 - \cos^2(2x)\,dx = \frac{1}{4}\int_0^\pi 1 - \frac{1+\cos(4x)}{2}dx$ second correct use of double-angle formula [1]

$= \frac{1}{8}\int_o^\pi 1 + \cos(4x)\,dx$

$= \frac{1}{8}\left[x + \frac{1}{4}\sin(4x)\right]_0^\pi$ correct integral found [1]

$= \frac{1}{8}\left[\left(\pi + \frac{1}{4}\sin(4\pi)\right) - \left(0 + \frac{1}{4}\sin(0)\right)\right]$

$= \frac{1}{8}(\pi + 0 - 0)$

$= \frac{\pi}{8}$ answer [1]

Question 10a

$x = t^3 \Rightarrow t = x^{\frac{1}{3}}$

$y = \log_e(t) = \log_e(x^{\frac{1}{3}})$ answer [1]

Question 10b

$\boldsymbol{v}(t) = \frac{d}{dx}\left(\boldsymbol{r}(t)\right)$

$= 3t^2\boldsymbol{i} + \frac{1}{t}\boldsymbol{j}$ derivative [1]

$speed = |\boldsymbol{v}(t)|$

$= \sqrt{(3t^2)^2 + \left(\frac{1}{t}\right)^2}$ [1]

$= \sqrt{9t^4 + \frac{1}{t^2}}$ answer [1]

Question 10c

Speed is a minimum when $\frac{d}{dx}(|\boldsymbol{v}(t)|) = 0$

stating derivative of speed should be zero for speed to be at a minimum [1]

$\frac{d}{dx}\left(\sqrt{9t^4 + \frac{1}{t^2}}\right) = \frac{1}{2}(9t^4 + t^{-2})^{-\frac{1}{2}}(36t^3 - 2t^{-3}) = 0$ evaluation of derivative [1]

$\left(9t^4 + \frac{1}{t^2}\right)^{-\frac{1}{2}} \neq 0$ due to the negative power

$36t^3 - 2t^{-3} = 0$ by the null factor theorem.

$36t^3 = \frac{2}{t^3}$

$t^6 = \frac{1}{18}$

$\therefore t = \left(\frac{1}{18}\right)^{\frac{1}{6}}$ answer [1]