B

Free Exam
for 2006-15 VCE study design

Engage Education Foundation

Units 3 and 4 Specialist Maths: Exam 2

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Any questions?

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Section A – Multiple-choice questions

Question 1
The correct answer is A.

It is clear that -2 is the only real valued solution. Hence, the remaining two solutions must be complex conjugates of one another; it suffices to find one. We can express z^3 in polar form as $z^3 = 8\ \text{cis}(\pi)$. By De Moivre's theorem, $z = 2\ \text{cis}\left(\frac{\pi}{3}\right) = 1 + \sqrt{3}\,i$ is a solution. Therefore, $z = 1 - \sqrt{3}\,i$ is a solution.

Question 2
The correct answer is D.

Shaded area is contained between circles of radius 3 (inclusive) and 4 (exclusive), below the perpendicular bisector of the line connecting $z = -i$ and $z = 1$.

Question 3
The correct answer is B.

Asymptotes occur when $2x + \frac{\pi}{4} = k\pi$, for any integer k. Rearranging in terms of x gives $x = \frac{(4k-1)\pi}{8}$. Substituting appropriate values of k gives the desired result.

Question 4
The correct answer is D.

Question 5
The correct answer is C.

Radius is 50mm, so the height is 120mm ((5,12,13) is a Pythagorean triple). As radius and height are in equal proportion at any depth, $r = \frac{12}{5}h$. We can then express V in terms of h only as $V = \frac{1}{3}\pi\left(\frac{12}{5}h\right)^2 h = \frac{48}{25}\pi h^3$. Then, by using the chain rule, $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = \frac{144}{25}\pi h^2 \times \frac{5}{4\pi} = \frac{36h^2}{5}$

Question 6
The correct answer is B.

$\frac{dy}{dx} = Ake^{kx}$ and $\frac{d^2y}{dx^2} = Ak^2e^{kx}$. Therefore we need to solve $Ak^2e^{kx} = -4Ae^{kx}(k+1)$. Dividing through by common terms and rearranging gives us the quadratic equation $k^2 + 4k + 4 = 0$, which has the unique solution $k = -2$.

Question 7
The correct answer is B.

The angle subtended by the circumference at any point on the circle (except A and C) is a right angle. So $\overline{CB} = 2\text{r}\sin\alpha$. Also ∠BCA is $\frac{\pi}{2} - \alpha$. As $\overline{DB}$ is perpendicular to the circumference, $\sin\left(\frac{\pi}{2} - \alpha\right) = \frac{\frac{1}{2}\overline{DB}}{\overline{CB}}$, so

$$\overline{DB} = 2\overline{CB}\sin\left(\frac{\pi}{2} - \alpha\right) = 2(2r\sin\alpha)\cos\alpha = 2r\,(2\sin\alpha\cos\alpha) = 2r\sin(2\alpha)$$

Question 8
The correct answer is B.

$|\boldsymbol{b}| = \sqrt{1^2 + 2^2 + 1^2}$, so $\hat{\boldsymbol{b}} = \frac{1}{\sqrt{6}}(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$.

$(\boldsymbol{a} \cdot \boldsymbol{b})\frac{\hat{\boldsymbol{b}}}{|\boldsymbol{b}|} = (3 - 8 - 1)\frac{1}{6}\boldsymbol{b} = -\boldsymbol{b}$

Question 9
The correct answer is C.

If $\cos\theta = \frac{2}{7}$, then $\sin\theta = \frac{\sqrt{7^2-2^2}}{7} = \frac{3\sqrt{5}}{7}$.Evaluate $\tan^{-1}(\sin\theta)$ using a calculator.

Question 10
The correct answer is A.

Make the observation that $f(x)$ can be expressed as $f(x) = \frac{\frac{d}{dx}(2x+3)}{(2x+3)^2+1}$ which looks very similar to the derivative of the inverse tangent function. Making the substitution $u = 2x + 3$ and yields the result $\int f(x)dx = \tan^{-1}(2x+3) + c$

Question 11
The correct answer is D.

Acceleration down plane is $mg\sin\theta = 6g\sin 25° = 24.9\ N$ down the plane. The maximum friction is $\mu\, mg\cos\theta = 3m\cos 25° = 26.6\ N$ up the plane. $26.6 > 24.9$. Hence the friction is $24.9N$ up the plane.

Question 12
The correct answer is A.

Given the shape and that the intersection of the asymptotes of the hyperbola is $(3,1)$, the equation must be of the form $\frac{(x-3)^2}{a^2} - \frac{(y-1)^2}{b^2} = 1$. As $\theta = \frac{\pi}{3}$, we can find the acute angle made by each asymptote and the x-axis which also turns out to be $\frac{\pi}{3}$. Therefore, the gradients of the asymptotes are $\pm\frac{b}{a} = \pm\tan\frac{\pi}{3} = \pm\sqrt{3}$. Hence possible values for a^2 and b^2 are 1 and 3, respectively.

Question 13
The correct answer is E.

We need to evaluate an integral of the form $\int_0^2 \pi \cdot \left(\frac{\pi}{2}\right)^2 dx - \int_0^2 \pi x(y)^2 dx = \int_0^2 \pi\left(\left(\frac{\pi}{2}\right)^2 - \left(x(y)\right)^2\right)dx$ (n.b. $x(y)$ denotes x as a function of y, i.e. . $(y) = \frac{\cos^{-1}(1-y)}{2}$): we find the volume of a cylinder and then 'hollow it out' by subtracting the volume we don't need. Evaluate using a calculator.

Question 14
The correct answer is D.

Question 15
The correct answer is A.

Look at the intercepts.

Question 16
The correct answer is B.

$\frac{dy}{dx} = 2\cos(2x) + 2\sin(2x)$ and $\frac{d^2y}{dx^2} = -4\sin(2x) + 4\cos(2x)$.

Question 17
The correct answer is C.

$4\cos 60° = 2N$, and $4\sin 60° = 2\sqrt{3}N$. Therefore $|F_{net}| = \sqrt{\left(5 - 2\sqrt{3}\right)^2 + (3-2)^2} = 1.83N$

Question 18
The correct answer is C.

Solve $\int_0^k 20dt = \int_0^k 5\sqrt{t}dt$ for k. Evaluating integrals and factorizing gives $10k\left(2 - \frac{1}{3}\sqrt{k}\right) = 0$, which has non-trivial solution $\sqrt{k} = 6 \Rightarrow k = 36$

Question 19
The correct answer is C.

Look at key features of the slope field; there are exactly two lines along which the gradient is zero (corresponds to a quadratic). More decisively, the gradient is independent of x; along any line $y = c$, where c is a constant, the gradient is the same; C is the only option where $\frac{dy}{dx}$ is independent of x.

Question 20
The correct answer is D.

$\dot{\boldsymbol{r}}(t) = 2\cos(t)\,\boldsymbol{i} - 2\sin(t)\,\boldsymbol{j} - \frac{\pi^2}{\left(t+\frac{\pi}{2}\right)^2}\boldsymbol{k}$. Evaluating $|\dot{\boldsymbol{r}}(0)|$ gives $\sqrt{20} = 2\sqrt{5}$

Question 21
The correct answer is C.

Simple evaluation.

Question 22
The correct answer is A.

Use Newton's 2nd Law, $F = ma$. Note speed is a positive quantity by definition (magnitude of velocity).

Section B – Short-answer questions

Marks allocated are indicated by a number in square brackets, for example, [1] indicates that the line is worth one mark.

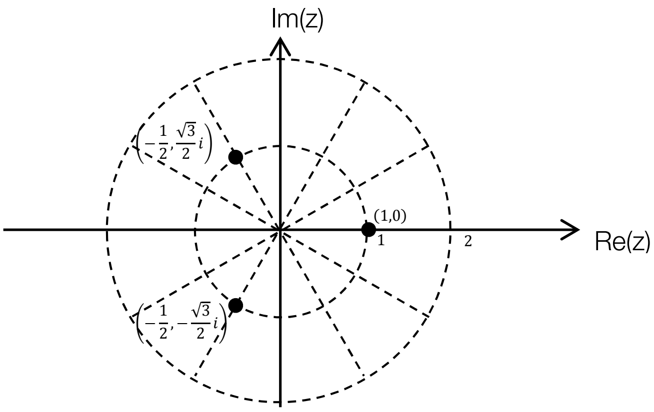
Question 1a i

Find the complex polar representation of 1, i.e. $1 = \text{cis}(0 + 2n\pi)$, where n is an integer.

So $z^3 = \text{cis}(2n\pi)$, and by De Moivre's theorem, $z = \text{cis}(\frac{2n\pi}{3})$. [1]

So the possible values of z are $\text{cis}(0) = 1$, $\text{cis}\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\text{cis}\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. [1]

Question 1a ii



[1] for correct markings, [1] for correct labels.

Question 1b i

Observe that $i^3 = -i$, so $(-i)^3 = i$. [1]

Now if $w^3 = 1$, then $(-iw)^3 = (-i^3)(w^3) = i$. Therefore, $k = -i$. [1]

Question 1b ii

The geometric interpretation of multiplication by i is a rotation of $\frac{\pi}{2}$ radians counter clockwise. So multiplication by k corresponds to rotation $\frac{3\pi}{2}$ counter clockwise. [1]

Therefore, the solutions are $\text{cis}\left(\frac{3\pi}{2}\right)$, $\text{cis}\left(\frac{\pi}{6}\right)$ and $\text{cis}\left(\frac{5\pi}{6}\right)$. [1]

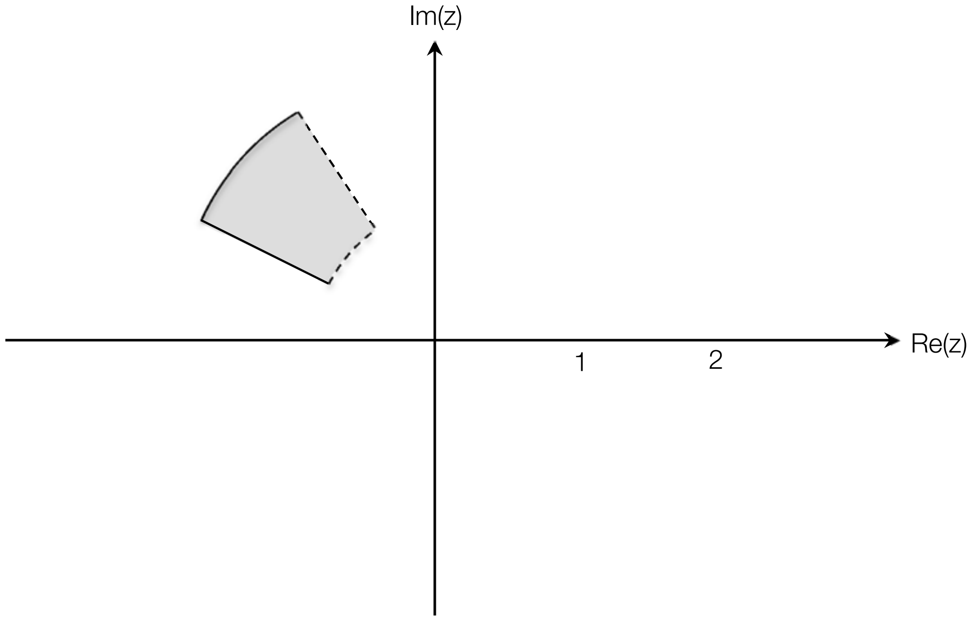
Question 1c

$z_1 = \text{cis}\left(\frac{2\pi}{3}\right)$ and $z_2 = \text{cis}\left(\frac{5\pi}{6}\right)$. [1]

$|z - z_1| = |z - z_2|$ describes the perpendicular bisector of the line joining these points. As both z_1 and z_2 have the same magnitude, the bisector must pass through the origin (n.b the bisector of a chord passes through the centre of the circle). The angle halfway between $\frac{4\pi}{6}$ and $\frac{5\pi}{6}$ is $\frac{9\pi}{12} = \frac{3\pi}{4}$, so the Cartersian equation is $y = -x$. [1]

Question 1d

[2] for correct shape, [1] for correct boundaries.



The first part of the set describes the set of points whose distance from iz_1 is strictly less that the distance from $-iz_1 = i^3 z_1$: i.e. the set of all points below the line joining z_1 to the origin. Similarly, the second part describes the set of points whose distance from iz_2 is greater than or equal to the distance from $i^3 z_2$: i.e. the set of points above the line joining z_2 to the origin. The third part requires that the magnitude of z is less than or equal to 2, but strictly greater than 1.

Question 2a

$f'(x) = \frac{-3}{1+x^2} + \frac{(x^2+1)(2)-(2x)(2x)}{(x^2+1)^2} + x^2 + 1$ applying appropriate rules [1]

$f'(x) = -\frac{1}{x^2+1} - \frac{4x^2}{(x^2+1)^2} + \frac{x^4+2x^2+1}{x^2+1}$

$$f'(x) = \frac{(x^4 + 2x^2)(x^2 + 1) - 4x^2}{(x^2 + 1)^2}$$

$f'(x) = \frac{x^6+2x^4+x^4+2x^2-4x^2}{(x^2+1)^2}$ correct manipulation [1]

$f'(x) = \frac{x^2(x^4+3x^2-2)}{(x^2+1)^2}$ correct answer [1]

Question 2b

$f'(x) = 0 \Leftrightarrow -x^2(x^4 + 3x^2 - 2) = 0$ as the denominator of $f'(x)$ is always positive

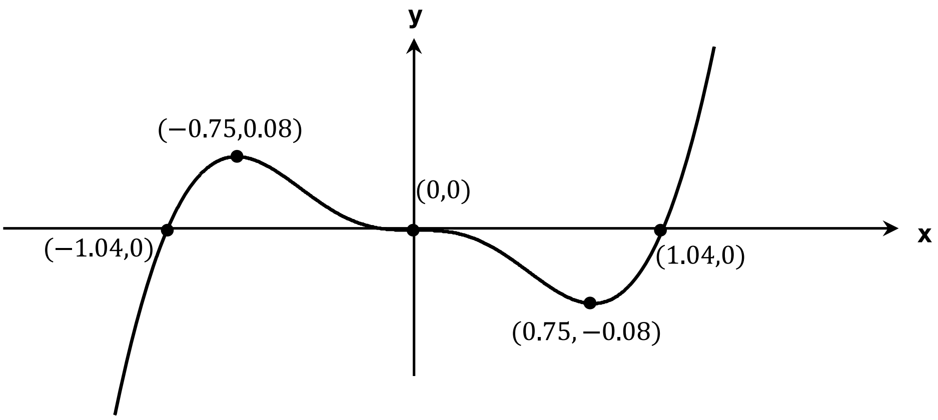
Let $u = x^2$, then $u(u^2 + 3u - 2) = 0$ observing that the function is a cubic in x^2 [1]

$u = \frac{-3\pm\sqrt{9+8}}{2}$ solving the quadratic term of the cubic [1]

$x = \sqrt{u} = \pm\sqrt{\frac{-3+\sqrt{17}}{2}}$ solving for x, requiring x to be real-valued [1]

$x = 0,\ x = \sqrt{\frac{-3+\sqrt{17}}{2}}\ , x = -\sqrt{\frac{-3+\sqrt{17}}{2}}$ listing solutions of $f'(x) = 0$ [1]

Question 2b



[2 for correct shape, 1 for correct intercepts, 1 for correct turning points]

Question 2c

$\int f(x)dx = -3\int \tan^{-1} x \ \ dx \ + \int \frac{2x}{x^2+1} dx + \int \frac{x^3}{3} + x\, dx$

$-3\int \tan^{-1} x \ \ dx = -3(x \tan^{-1} x - \frac{1}{2}\log_e(x^2+1))$ [1]

$\int \frac{2x}{x^2+1} dx = \int \frac{1}{u} du = \log_e u = \log_e(x^2+1)$ substituting $u = x^2 + 1$ [1]

$\int \frac{x^3}{3} + x\, dx = \frac{x^4}{12} + \frac{x^2}{2}$

Therefore,

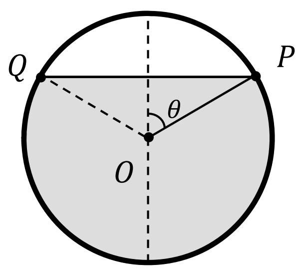
$\int f(x)dx = -3\left(x \tan^{-1} x - \frac{1}{2}\log_e(x^2+1)\right) + \log_e(x^2+1) + \frac{x^4}{12} + \frac{x^2}{2}$

$= -3x \tan^{-1} x + \frac{5}{2}\log_e(x^2+1) + \frac{x^4}{12} + \frac{x^2}{2}$ [1]

Question 3a

$V = 20m^3$, and the surface area of the circle is $\pi\, m^2$. Therefore, $L = \frac{20}{\pi}$ m [1]

Question 3b i



Area of the sector subtended by the angle$\angle POQ$ is given by $\frac{2\theta}{2\pi}\pi r^2 = \theta$ [1/2]. The area of the triangle $\triangle POQ$ is given by $\sin\theta \times \cos\theta = \frac{1}{2}\sin 2\theta$ [1/2]. So the area of the unshaded segment is given by $\theta - \frac{1}{2}\sin(2\theta)\, m^2$ [1]

Question 3b ii

The total volume is $20m^3$, and the unfilled volume is $\left(\theta - \frac{1}{2}\sin(2\theta)\, m^2\right) \times L$. [1] So the amount of water in the tank as a function of θ is $V = 20 - \frac{L}{2}(2\theta - \sin(2\theta))$, where $0 \le \theta \le \pi$, as these are the only values θ can physically take. [1]

Question 3b iii

Take the derivative with respect to time of both sides of the equation in part ii.

$$\frac{dV}{dt} = \frac{d}{dt}\left(20 - \frac{L}{2}(2\theta - \sin(2\theta))\right) = \frac{d\theta}{dt}\frac{d}{d\theta}\left(20 - \frac{10}{\pi}(2\theta - \sin(2\theta))\right) = \frac{d\theta}{dt}\left(-\frac{20}{\pi}(1 - \cos(2\theta))\right) \text{ [2]}$$

Rearranging in terms of $\frac{d\theta}{dt}$, given that $\frac{dV}{dt} = -2$:

$$\frac{d\theta}{dt} = \frac{\pi}{10(1-\cos(2\theta)\,)} \text{ [1]}$$

Question 3c

$$\theta(0.1) = \theta(0) + \Delta t\left(\frac{d\theta}{dt}\right)_{t=0} = 0 + 0.1 \times \frac{\pi}{10} = \frac{\pi}{100} \quad \text{[1]}$$

$$\theta(0.2) = \theta(0.1) + \Delta t\left(\frac{d\theta}{dt}\right)_{t=0.1} = \frac{\pi}{100} + 0.1 \times \frac{\pi}{10(1-\sin(0.2))} \quad \text{[1]}$$

$$\theta(0.3) = \theta(0.2) + \Delta t\left(\frac{d\theta}{dt}\right)_{t=0.2} = \left(\frac{\pi}{100} + 0.1 \times \frac{\pi}{10(1-\sin(0.2))}\right) + 0.1 \times \frac{\pi}{10(1-\sin(0.4))} \approx 0.122\ rad/s \text{ [1]}$$

Question 4a

$$|\boldsymbol{r}(t)| = \sqrt{(2t)^2 + \left(2e^{-\frac{t^2}{10}}\cos\frac{\pi t}{5}\right)^2 + \left(2e^{-\frac{t^2}{10}}\sin\frac{\pi t}{5}\right)^2}$$ displacement is given by the $|r(t)|$ [1]

$$= \sqrt{4t^2 + \left(4e^{-\frac{t^2}{5}}\right)\left(\cos\left(\frac{\pi t}{5}\right)^2 + \sin\left(\frac{\pi t}{5}\right)^2\right)}$$ rearranging

$$= 2\sqrt{t^2 + e^{-\frac{t^2}{5}}}$$ [1]

Question 4b

$$\dot{\boldsymbol{r}}(t) = \frac{d}{dt}(2t)\boldsymbol{i} + \frac{d}{dt}\left(2e^{-\frac{t^2}{10}}\cos\frac{\pi t}{5}\right)\boldsymbol{j} + \frac{d}{dt}\left(2e^{-\frac{t^2}{10}}\sin\frac{\pi t}{5}\right)\boldsymbol{k}$$

$$\frac{d}{dt}(2t) = 2$$

$$\frac{d}{dt}\left(e^{-\frac{t^2}{10}}\cos\frac{\pi t}{5}\right) = \frac{d}{dt}\left(e^{-\frac{t^2}{10}}\right)\cos\frac{\pi t}{5} + \frac{d}{dt}\left(\cos\frac{\pi t}{5}\right)e^{-\frac{t^2}{10}} = -\frac{te^{-\frac{t^2}{10}}}{5}\cos\frac{\pi t}{5} - \frac{\pi}{5}\sin\frac{\pi t}{5}e^{-\frac{t^2}{10}} \quad \text{[1]}$$

$$\frac{d}{dt}\left(e^{-\frac{t^2}{10}}\sin\frac{\pi t}{5}\right) = \frac{d}{dt}\left(e^{-\frac{t^2}{10}}\right)\sin\frac{\pi t}{5} + \frac{d}{dt}\left(\sin\frac{\pi t}{5}\right)e^{-\frac{t^2}{10}} = -\frac{te^{-\frac{t^2}{10}}}{5}\sin\frac{\pi t}{5} + \frac{\pi}{5}\cos\frac{\pi t}{5}e^{-\frac{t^2}{10}} \quad \text{[1]}$$

$$\dot{\boldsymbol{r}}(t) = 2\boldsymbol{i} - \frac{2e^{-\frac{t^2}{10}}}{5}\left(t\cos\frac{\pi t}{5} + \pi\sin\frac{\pi t}{5}\right)\boldsymbol{j} + \frac{2e^{-\frac{t^2}{10}}}{5}\left(-t\sin\frac{\pi t}{5} + \pi\cos\frac{\pi t}{5}\right)\boldsymbol{k} \quad \text{[1]}$$

Question 4c

$|\boldsymbol{r}(5)| = 2\sqrt{25 + e^{-5}} \approx 10m$ [1/2]

$\dot{\boldsymbol{r}}(5)=2\boldsymbol{i} - \frac{2e^{-\frac{5}{2}}}{5}(-5)\,\boldsymbol{j} + \frac{2e^{-\frac{5}{2}}}{5}(-\pi)\boldsymbol{k}$ [1/2]

$|\dot{\boldsymbol{r}}(5)| = \sqrt{4 + 4e^{-5} + \frac{4\pi}{25}e^{-5}} \approx 2\ m/s$ [1]

Question 5a

The gradient of the ramp is given by $\frac{dy}{dx} = \frac{x}{2}$, so $\tan(\theta) = \frac{x}{2}$ [1]

Then, the magnitude of the normal is given by $|F_n| = mg\cos\theta = 49\frac{2}{\sqrt{x^2+4}}$ [1]

We now need to resolve this into components parallel to the axes. A little geometric manipulation yields that the horizontal component is $-F_n \sin\theta$, and the vertical component is $F_n \cos\theta$ [1]

Therefore, $F_{normal} = -49\frac{4}{x^2+4}\boldsymbol{i} + 49\frac{2x}{x^2+4}\boldsymbol{j} = \frac{196}{x^2+4}\boldsymbol{i} + \frac{98x}{x^2+4}\boldsymbol{j}$ [1]

Question 5b i

As before, the gradient of the ramp at a point is given by $\frac{dy}{dx} = \frac{x}{2}$, and $\tan(\theta) = \frac{x}{2}$. The magnitude of the force tangent to the ramp (and therefore parallel to the direction of acceleration of the mass), is $|F| = mg\sin\theta = 49\frac{2x}{\sqrt{x^2+4}}$ [1]

Then $a = \frac{19.6x}{\sqrt{x^2+4}}$ by Newton's second law. [1]

Question 5b ii

As $a = \frac{d}{dx}\frac{1}{2}v^2$, we integrate both sides with respect to x from $x = k$ to $x = 0$ (we reverse the limits to account for the direction of acceleration) and solve for v. [1] We will have to do a u-substitution, so choose $u = x^2 + 4$, then $\frac{du}{dx} = 2x$ [1]

$\frac{1}{2}v^2 = \int_k^0 \frac{19.6x}{\sqrt{x^2+4}}dx = \int_{k^2+4}^4 \frac{19.6x}{\sqrt{u}}\frac{1}{2x}du = 9.8\left[\frac{1}{2}\sqrt{u}\right]_{\mathrm{k+4}}^4 = 4.9\left(\sqrt{k^2+4} - 2\right)$ [2]

Therefore,

$v = \sqrt{9.8\left(\sqrt{k^2+4} - 2\right)}$ [1]

We ignore the negative root as speed must be positive.

Question 5b iii

Evaluating v at $a = 2$ gives $v = \sqrt{9.8\left(2\sqrt{2} - 2\right)} = 2.84\ m/s$ [1]